

# A survey of Mathematical Methods for Indoor Localization

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**Abstract**—This document provides a survey of the mathematical methods currently used for position estimation in indoor local positioning systems (LPS), particularly those based on radiofrequency signals. The techniques are grouped into four categories: geometry-based methods, minimization of the cost function, fingerprinting, and Bayesian techniques. Comments on the applicability, requirements, and immunity to nonline-of-sight (NLOS) propagation of the signals of each method are provided.

## I. INTRODUCTION

An intense research work by the scientific community is carried today to design and build localization systems that can operate in indoor environments and achieve a degree of precision, reliability and cost comparable to the well known GPS system. The availability of such LPS (local positioning systems) will permit numerous advances in the disciplines of location-aware and pervasive computing, ambient intelligence, and facilitate the deployment of location-based services (LBS) [1]. Of the many technological possibilities that have been considered for LPS, like ultrasonic, infrared and artificial vision, radiofrequency based systems predominate today, due to their availability and low cost. Prototypes based on wifi, zigbee, bluetooth, ultrawideband radio, RFID and mobile telephony, as well as proprietary solutions have been described in the literature (see recent surveys in [2] and [3]).

While most existing literature focuses on the physical design and evaluation of LPS systems, this communication aims to present a comprehensive study of the mathematical methods used for localization, which are the backbone of all existing systems. We hope that this document can be used as a primer by new researchers incorporating to the area of local positioning systems.

### A. Local Positioning Systems

A local positioning system consists in a set of base stations (BS) placed at known locations in the displacement area, and a mobile station (MS) which is carried by the person or object to be located. During the operation of the LPS, signals are exchanged between the BS and the MS which enable the localization of the latter. When the signals are transmitted from the MS to the BS we speak of a centralized LPS, in which the infrastructure is aware of the presence and location of the user; in the opposite scheme, the MS utilizes signals transmitted from the infrastructure to compute his location on his own.

Location methods are further classified by the measurable quantities obtained from the transmitted signals. Techniques based on the measurement of the time of arrival (TOA) of such signals are generically named trilateration. If absolute measurements of the TOA are available, we speak of spherical trilateration, and, when only relative time differences (TDOA) can be physically measured (due to lack of synchronization between the MS and the BS), we speak of hyperbolic trilateration. Triangulation methods are employed when the angle of arrival (AOA) of the signals transmitted from MS to BS can be measured. Very often in indoor positioning systems based on RF, only the received signal strength (RSS) is available for localization (for example, the IEEE 802.15.4 standard supports RSS-based ranging). Finally, some LPS use a measurement of the quality of the RF transmission in digital communication channels, like the link quality indicator LQI or the bit error rate, BER, or even simple connectivity of the MS to a given BS.

We will tacitly assume that the mobile station has access to enough base stations at known positions in the infrastructure so that it can calculate its position individually. Alternatively, a set of MS may have to localize themselves cooperatively, since individually they don't have access to enough BS; this is termed cooperative or multi-hop localization, and is of great importance in wireless sensor networks. An excellent review of cooperative localization techniques can be found in [4]. In this communication we will deal with the case of individual, or one-hop, localization.

### B. Mathematical background

Suppose that the MS of an LPS receives a set of signals arriving from  $n$  base stations placed at known locations  $\mathbf{x}_i$  ( $i = 1, \dots, n$ ) in the displacement environment, and from them it obtains a set of measurements  $\mathbf{r} = \{r_i\}$ , with the goal of computing its current location  $\mathbf{x}$ . The relationship between the measurable variables and the unknown position can in general be written as [5]:

$$\mathbf{r} = \mathbf{h}(\mathbf{x}) + \mathbf{e}, \quad (1)$$

where  $\mathbf{h}$  is a function which implicitly contains the positions of the base stations ( $\mathbf{x}_i$ ), and  $\mathbf{e}$  is the error affecting the measurement, with a probability density function  $p_e(\mathbf{e})$ . The functional dependencies  $\mathbf{h}(\mathbf{x})$  and  $p_e(\mathbf{e})$  completely characterize the measurement process [6]. The maximum likelihood

(ML) estimation of the position is the one which maximizes the conditional probability:

$$\hat{\mathbf{x}} = \arg \max \{p(\mathbf{r}|\mathbf{x})\}. \quad (2)$$

Theoretical bounds can be established for the ultimately attainable precision of any estimation method, by using the Fisher information matrix and the Cramer-Rao lower bound (CRLB). The CRLB has been computed in the literature for most estimation techniques ([6] and references therein).

The physical laws of the propagation of signals in indoor environments pose a problem for all localization methods. The signal propagating through the line-of-sight (LOS) from BS to MS might be blocked by an obstacle, but it is possible that a reflection of it can still reach the receiver by an alternate route. This is termed non line-of-sight propagation or NLOS. For LPS based on the measurement of the TOA, NLOS causes a systematic overestimate of the actual range between BS and MS, and leads to biased estimations of position. Likewise, the arrival of interfering copies of the emitted signal produced by multipath deteriorates the estimations of the TOA or the RSS at the receiver, when compared with propagation in free space. In a similar way, the amplitude of RF signals are attenuated by obstacles in the direct way from BS to MS, deteriorating position estimations based on the signal strength. Since these problems are intrinsic to indoor infrastructures (due to the presence of walls, doors, furniture, presence of people), location methods must be designed to mitigate the impact of NLOS situations, an issue which we will discuss as we describe the methods themselves in the following sections.

In this paper we propose a classification of localization methods into four categories: geometry-based methods, minimization of the cost function, fingerprinting, and Bayesian techniques. The next sections describe each of these categories in detail.

## II. GEOMETRY-BASED METHODS

Geometric methods are used when ranges, difference of ranges, or angles between the BS and MS can be measured with relatively small error. In this case, the set of equations given by  $\mathbf{h}(\mathbf{x})$  is easily established by simple algebraic relationships. For example, for TOA, the range between the MS and the  $i$ -th BS is given as:

$$r_i^{\text{TOA}}(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_i\| + e_i, \quad i = 1, \dots, n, \quad (3)$$

where  $\mathbf{e}$  is assumed to follow a zero-mean Gaussian distribution,  $p_e(\mathbf{e}) = N(0, \Sigma)$ . The relationships for TDOA and AOA are obtained similarly from geometrical considerations, while the equation for RSS can be obtained through a propagation model, as we will see in section III. Even in this simple situation, the resulting set of equations is nonlinear and, in the general case, overdetermined, and therefore, it cannot be solved in an exact closed way. However, both approximate closed-form and exact iterative solutions can be provided.

One simple linearization scheme expands eq. 3 and groups together the nonlinear terms in an additional variable. Through

a straightforward manipulation we arrive to the following matricial form:

$$\mathbf{R} = \mathbf{A}\mathbf{X}, \quad (4)$$

or

$$\begin{bmatrix} r_1^2 - \|\mathbf{x}_1\|^2 \\ \dots \\ r_n^2 - \|\mathbf{x}_n\|^2 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ \dots & \dots & \dots \\ x_n & y_n & 1 \end{bmatrix} \begin{bmatrix} -2x \\ -2y \\ \|\mathbf{x}\|^2 \end{bmatrix}, \quad (5)$$

where localization in the plane  $\mathbf{x} = (x, y)$  is assumed. The least-squares solution of eq. 5 is found by the pseudoinverse:

$$\hat{\mathbf{X}} = (\mathbf{A}^T \Sigma_2^{-1} \mathbf{A})^{-1} \mathbf{A}^T \Sigma_2^{-1} \mathbf{Z}, \quad (6)$$

where  $\Sigma_2$  is the covariance matrix error of the set of measurements  $\{r_i^2\}$ . Although this method provides a closed form solution, the results are not optimal, since the third element of vector  $\mathbf{X}$  is not an independent variable. In the case that the covariance matrix of the error is not known, it can be estimated along with the position by an iteratively reweighted least squares (IRLS) method [7], provided that we have many different measurements from every BS.

Another simple linearization technique [8] consists in subtracting the first equation of eq. 5 from the remaining others; obtaining a linear system of equations of the general form given by eq. 4, with

$$\mathbf{R} = \begin{bmatrix} r_2^2 - r_1^2 + \|\mathbf{x}_1\|^2 - \|\mathbf{x}_2\|^2 \\ \dots \\ r_n^2 - r_1^2 + \|\mathbf{x}_1\|^2 - \|\mathbf{x}_n\|^2 \end{bmatrix}, \quad (7)$$

$$\mathbf{A} = \begin{bmatrix} -2(x_2 - x_1) & -2(y_2 - y_1) \\ \dots & \dots \\ -2(x_n - x_1) & -2(y_n - y_1) \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}. \quad (8)$$

Although these equations are truly linear in the MS coordinates, we have discarded information during the linearization process, and therefore this method does not provide optimum estimations either.

Other techniques include casting the set of trilateration equations into a constrained minimization problem, as in [9], the use of multidimensional scaling [10], and subspace decomposition techniques [11]. Equivalent closed form solutions for TDOA [12] [13] and AOA [14] [15] based localization also exist. In conclusion, closed form solutions, while convenient, do not provide maximum likelihood position estimations, and will in general be biased, even for the case of unbiased Gaussian errors in the measurements  $\{r_i\}$ .

Maximum likelihood estimations can be obtained from the nonlinear equations of geometry-based LPS by linearization through a Taylor expansion [16]. A linearized version of the location equations around a position estimate  $\mathbf{x}_0$  is obtained by using the Jacobian of  $\mathbf{h}(\mathbf{x})$ :

$$\mathbf{h}(\mathbf{x}) \simeq \mathbf{h}(\mathbf{x}_0) + \mathbf{J}_h(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0), \quad (9)$$

from which an improved position estimation  $\mathbf{x}$  is produced iteratively as:

$$\mathbf{x} = \mathbf{x}_0 + (\mathbf{J}_h^T(\mathbf{x}_0) \Sigma^{-1} \mathbf{J}_h(\mathbf{x}_0))^{-1} \mathbf{J}_h^T(\mathbf{x}_0) \Sigma^{-1} (\mathbf{r} - \mathbf{h}(\mathbf{x}_0)),$$

where  $\Sigma$  is the covariance error of the measurement vector  $\mathbf{r}$ . Taylor linearization works well in GPS positioning, since the variations of pseudoranges caused by displacement of the user on the surface of the Earth are small compared to the distances to the satellites. For indoor cases, where this is not necessarily the case, Taylor linearization might be prone to convergence errors and sensitivity to initial conditions. A good position seed for the Taylor expansion can be provided by the closed form solutions described above.

Implicit in the equations  $\mathbf{r} = \mathbf{h}(\mathbf{x})$  there is a functional dependence with the geometric arrangement of the base stations placed at positions  $\{\mathbf{x}_i\}$ . In determinate circumstances, this may lead to problems computing the pseudoinverse in eq. 6 or the Jacobian in eq. 9, and physically, to large errors of the position estimation  $\sigma_{\mathbf{x}}^2$  relative to estimations of the measured quantities themselves  $\sigma_{\mathbf{r}}^2$ . This phenomenon of amplification of the error  $\sigma_{\mathbf{x}}^2/\sigma_{\mathbf{r}}^2$  is called the dilution of precision (DOP), and has been studied extensively in the GPS literature [17]. When deploying the base stations in the infrastructure, attention must be taken to guarantee low values of DOP in the working area [18].

Geometry-based localization methods are computationally efficient, but not particularly well suited for detection of outlier data  $r_i$  caused by NLOS. Only in the case that the outlier data are few and sufficient redundancy exists (typically, in ultrasonic and ultrawideband radio based LPS), it is feasible to detect NLOS instances. Hypothesis testing is used, with the null hypothesis  $H_0$  being LOS and the alternative  $H_1$ , the occurrence of NLOS in a given measurement. An increase of the variance error of time records of measured ranges might be indicative of NLOS [19]. Techniques for NLOS correction take advantage of the redundancy and self-consistency of the set of measurement data  $\mathbf{r}$ ; for example, by examining the residue between the empirical measurements  $\mathbf{r}$  and those computed back from the estimated position  $\hat{\mathbf{x}}$  [20]. Measurements with large residues correspond likely to outlier data caused by NLOS, which can be excluded, and the process repeated for an improved estimation. The parity space, which is a linear transformation of the residual errors of the measurements  $\mathbf{r}$  into a form which permits the identification of deviation errors caused by NLOS [21] can also be used. This method is designed to work with linearized versions of the  $\mathbf{h}(\mathbf{x})$  equations (such as those used in GPS positioning) and might not be applicable in indoor environments.

Other techniques for outlier rejection come from the robust statistics field. For example, multiple position estimations can be produced by exhaustively using small subsets of  $n_{\min}$  elements with the received ranges, and then a robust estimator like the median can be used for the final estimation of position [22]. Robustness is achieved at the expense of computational time, since the number of independent position estimations grows quickly with the measurement data (if  $n$  measurements  $\{r_i\}$  are available, the number of solutions to be computed is the binomial coefficient  $n!/(n - n_{\min})! n_{\min}!$ ).

### III. MINIMIZATION OF THE COST FUNCTION

Another approach for position estimation consists in finding the minimum of the following cost function:

$$V(\mathbf{x}) = \log p_e(\mathbf{r} - \mathbf{h}(\mathbf{x})), \quad (10)$$

since this will maximize  $p(\mathbf{r}|\mathbf{x})$ . Unlike the methods presented in the last section, direct minimization of the cost function is applicable to arbitrary error PDFs  $p_e(\mathbf{e})$ , and provides maximum likelihood position estimations [23]. Only in the case of zero-mean Gaussian noise with covariance matrix  $\Sigma$ , the cost function simplifies to:

$$V(\mathbf{x}) = (\mathbf{r} - \mathbf{h}(\mathbf{x}))^T \Sigma^{-1} (\mathbf{r} - \mathbf{h}(\mathbf{x})). \quad (11)$$

In general, the measurement errors need not be zero-mean, or Gaussian. Standard methods like Gauss-Newton or Levenberg-Marquardt can be used for minimization of  $V(\mathbf{x})$ . As in the case of Taylor linearization in section II, convergence problems arise for bad initialization values and also when the MS is close to a position where the arrangement of the BS causes high values of the DOP.

If the relationship  $\mathbf{r} = \mathbf{h}(\mathbf{x})$  can be parameterized, its free parameters, which depend on the environment, can be estimated alongside with the position by minimization of the cost function, provided there is enough redundancy in the set of measurements  $\{r_i\}$ . For the case of indoor localization through the signal strength, usually the following empirical dependence with range, known as path loss law, is used:

$$r_i^{\text{RSS}}(\mathbf{x}, \alpha_i) = \text{RSS}_0 - 10\alpha_i \log\|\mathbf{x} - \mathbf{x}_i\| + e_i, \quad (12)$$

where  $\text{RSS}_0$  is the power transmitted by the BS, and  $\alpha_i$  is the path loss exponent, which in general will be different for each base station. In these circumstances, an improved position estimate [24] can be produced by minimizing the cost function:

$$V(\mathbf{x}, \alpha_1, \dots, \alpha_n) = \sum_{i=1}^m \|r_i - r_i^{\text{RSS}}(\mathbf{x}, \alpha_i)\|,$$

for which the number of measurements  $m$  must be larger than the sum of the number of base stations  $n$  and the coordinates of the position to be estimated (i.e., several consecutive measurements to each BS must be obtained). More detailed models of the propagation characteristics of the signals can be used for improving the estimation of RSS. For example, the attenuation caused by the presence of walls between MS and BS can be incorporated to eq. 12. However, some researchers claim that these more complicated procedures do not lead to a significant increase in location accuracy [25].

Some protection against NLOS outliers is achieved by converting the unconstrained minimization problem in eq. 11 into a constrained one, as suggested in [26] for ultrasonic position estimation through measurement of the TOA. For this purpose, note that if  $r_i$  is the empirical TOA-based range of the signal from the MS to the  $i$ -th BS:

$$\|\mathbf{x} - \mathbf{x}_i\| - r_i \leq 0,$$

since multipath reflections always cause a propagation time larger than that corresponding to the actual LOS range between MS and BS.

As stated in the introduction, the elimination of the effects of outliers/NLOS conditions is especially problematic in indoor environments. For ultrasonic and UWB based LPS operating in not too complex environments, these errors can appear as isolated instances, so the methods described in the last two sections might be efficient. However, for other RF technology, the complexity of indoor propagation suggests that is better to consider NLOS as a natural part of the estimation problem, and accept that the probability error  $p_e$  in the measuring model of eq. 1 can be as large as the  $h(\mathbf{x})$  term itself. The next two classes of methods are designed to operate under these conditions.

#### IV. FINGERPRINT METHODS

Fingerprint methods are usually employed in LPS based on measurement of the received signal strength RSS, and consist in two phases. In the calibration stage, a test MS moves through a grid of sufficiently dense set of positions  $\{\mathbf{x}_j\}$  that cover the indoor environment and records the signal levels from the different base stations  $BS_i$ ,  $\{\rho_i(\mathbf{x}_j)\}$ , where  $i = 1, \dots, n$ . In the localization stage, the system reads a set of signals  $\{r_i\}$ , which is compared to the previously recorded set and the position that matches it most closely is chosen as the estimation of location. Fingerprint-based localization uses methods and techniques from machine learning (also called statistical learning) for finding the optimal estimate of position. Fingerprint methods make no formal distinction between LOS and NLOS measurements, and are therefore inherently robust to the occurrence of the latter.

One obvious choice for position estimation consists in picking the grid point  $\mathbf{x}_j$  which minimizes the Euclidean distance in the measurement variable space:

$$\hat{\mathbf{x}} = \arg \min \{z_j^2\}, \quad \text{with} \quad z_j^2 = \sum_{i=1}^n (r_i - \rho_i(\mathbf{x}_j))^2. \quad (13)$$

A simplified analysis of this situation is given in [27]. If we assume that the measured  $r_i$  follow Gaussian distributions with common variance,  $r_i = N(\rho_i, \sigma^2)$ , then  $z_j^2$  is  $\chi^2$  distributed, with  $n$  degrees of freedom, and it's central if  $\mathbf{x}_j$  is the true position of the MS, and non-central for the remaining points in the grid. Theoretically, the non-centrality parameter:

$$\lambda_j = \sum_i (\mathcal{E}\{r_i\} - \rho_i(\mathbf{x}_j))^2,$$

could be employed to choose the optimal location of the mobile, since it increases with the distance of  $\mathbf{x}_j$  to the real position. However, discriminating between central and non-central distributions proves difficult when the variance of the distribution of  $r_i$  is large (as it usually happens with signal strength based measurements in indoor environments). This places a limit in the benefits obtained by using a finer grid spacing for increased accuracy [27]. Following the same procedure, one could theoretically obtain the distribution error

of the estimated position  $\hat{\mathbf{x}}$ . The  $k$ -nearest neighbors (kNN) method is a variant of the Euclidean distance method, in which the position estimation is produced by averaging the position of the  $k$  position estimations with lower distances  $z_j$  in the signal space.

Bayesian inference methods [28] search for the maximum likelihood estimator of position (MLE), given as:

$$\hat{\mathbf{x}}_{\text{MLE}} = \arg \max \{p(r_1, \dots, r_n | \mathbf{x}_j)\}. \quad (14)$$

By Bayes' theorem [29], this is equivalent to finding the position  $\mathbf{x}_j$  which maximizes

$$p(\mathbf{x}_j | r_1, \dots, r_n) = \frac{p(r_1, \dots, r_n | \mathbf{x}_j)p(\mathbf{x}_j)}{p(r_1, \dots, r_n)}, \quad (15)$$

assuming that there is no a priori information about the position, so that all positions  $\mathbf{x}_j$  are equally likely. Computation of the conditional probability in eq. 15 is commonly simplified by supposing conditional independence of the measurement from all base stations:

$$p(r_1, \dots, r_n | \mathbf{x}_j) = p(r_1 | \mathbf{x}_j) \cdot p(r_2 | \mathbf{x}_j) \cdots p(r_n | \mathbf{x}_j), \quad (16)$$

and by assuming a normal PDF for  $p(r_i | \mathbf{x}_j) = N(\rho_i, \sigma^2)$ , as we did before. The position estimation can also be given as an area of confidence: for example a set of grid positions  $\{\mathbf{x}_j\}$ , such that  $\sum_j p(\mathbf{x}_j) > P_{th}$ , where  $P_{th}$  is a threshold probability. Machine learning methods are easily cast into classification problems, a feature which is useful for symbolic location and guidance of people, in which the interest lies in knowing in which area of a building (like a room or a corridor) the mobile user is located, more than in estimating his physical coordinates.

Other techniques from the field of Statistical Learning that have been used for indoor location include neural networks [30], decision trees [31] and support vector machines [32].

Fingerprint based methods usually produce the most accurate estimation of position in indoor environments. Indeed, some authors [33] have argued for the existence of fundamental limitations in the achievable accuracy of indoor signal-strength based localization, which can not be improved without new physical developments. Even if this is the case, fingerprint methods have some drawbacks that might impair their application in real situations. The calibration phase is time consuming, position estimations can only be produced in those areas where recorded data exists, and finally, in the event of modification of the indoor environment, their accuracy is greatly decreased. Interestingly, recent research has suggested that the presence of people in the operation area of the LPS can have a beneficial influence on the position estimation, by increasing the conditional independence in eq. 16 [34].

#### V. BAYESIAN METHODS

Bayesian methods for position estimation can be viewed as an extension of the Bayesian inference techniques of the last section [35]. The information about the position of the MS at time  $t$  is modelled as a probability distribution

$p(\mathbf{x}_t|r_1, r_2, \dots, r_t)$ , based in the set of all past sensor measurements  $r_1, r_2, \dots, r_t$  obtained up till time  $t$ .

The estimation of the position of the MS proceeds in two stages, called respectively the prediction and the correction steps. In the prediction step the *a priori* estimated position is extrapolated from previous data, without actually taking a measurement:

$$p^-(\mathbf{x}_t) = \int p(\mathbf{x}_t|\mathbf{x}_{t-1})p(\mathbf{x}_{t-1}) d\mathbf{x}, \quad (17)$$

where  $p(\mathbf{x}_t|\mathbf{x}_{t-1})$  is a motion model which represents our estimation of where the user will be in the next time interval from the current estimated position. This motion model can be produced by readings from a sensor carried by the MS (like odometers and inertial sensors in a mobile robot), or, in the case of a person, can consist of a region of possible displacements limited by the maximum velocity that the person can achieve, as well as features in the indoor environment like walls and doors that dictate which regions are accessible.

After the prediction step, the correction step matches the computed estimation of position with the set  $\mathbf{r}_t$  of sensor measurements collected in the time interval from  $t-1$  to  $t$ . Again, by Bayes rule, the posterior probability is found by multiplying the prior probability  $p^-(\mathbf{x}_t)$ , by the observation model  $p(\mathbf{r}_t|\mathbf{x}_{t-1})$ :

$$p(\mathbf{x}_t) = a_t p(\mathbf{r}_t|\mathbf{x}_{t-1})p^-(\mathbf{x}_{t-1}), \quad (18)$$

where  $a_t$  is a normalization constant such that the integral of the probability distribution over all possible positions in the displacement area is one.

The observation model  $p(\mathbf{r}|\mathbf{x})$  describes the probability of receiving measurement  $\mathbf{r}$  when the user is standing at position  $\mathbf{x}$ . As in fingerprint-based estimations, this model is generated during calibration from a large set of empirical measurements obtained at different locations covering the displacement area. The observation model can consist of histograms of the empirical measurands  $r_i$ , smoothed distributions fitted to the raw data, kernel-based models, etc [29]. Eqs. 17 and 18 are applied consecutively each time a new measurement is available to refine the current estimation of the user's position.

Bayesian estimation methods have a number of advantages for indoor positioning. They are robust to NLOS situations, which can be incorporated directly in the observation model:

$$p_e(e) = p_{\text{LOS}}N(0, \sigma_r^2) + (1 - p_{\text{LOS}})p_e^{\text{NLOS}}(e),$$

which can be compared to the hypothesis testing of section II. Depending on how much information we have about the probabilities of LOS and NLOS, and the PDFs of the error in both instances, effective mitigation of nonlinear-of-sight effects can be achieved [36].

Bayesian location methods are iterative, which permits to improve upon previous location estimations by processing many imprecise measurements. Bayesian localization permits to accommodate naturally measurements of different nature (for example, TOA and RSS), by simply multiplying their respective observation models. Moreover, no assumption on

the form of the PDF  $p(\mathbf{x})$  needs to be done, which permits great flexibility. For example, at a given time, the localization method may be unable to resolve in which room the MS is located, because of insufficient data. The Bayesian scheme permits to assume the mobile can be in two or more rooms, and compute their relative probabilities, waiting until this ambiguity is resolved when more measurement data are available.

Many different implementations exist in the generic frame of Bayesian localization. Kalman filters are the simplest case, in which the position error follows a normal distribution. Multihypothesis tracking is an extension of the Kalman filter, in which several different hypotheses for the MS are simultaneously considered. Likewise,  $p(\mathbf{x})$  can also be estimated over a discretized grid, or a Voronoi graph which topologically represents the displacement area as a set of nodes and edges [37]. This is well suited for the symbolic location described in section IV. An efficient implementation of Bayesian localization which has received much attention lately are particle filters. They can be visualized as an adaptive sampling of the position estimation PDF  $p(\mathbf{x})$  which focuses in the areas with higher position probability for improved efficiency [38]. In a particle filter, the probability distribution of the location is approximated by a discrete set of  $n_p$  points (the particles):

$$p(\mathbf{x}_t) \approx \{(\mathbf{x}_t^i, w_t^i)\}, \quad i = 1, \dots, n_p, \quad (19)$$

where  $\mathbf{x}_t^i$  is the location of the  $i$ -th particle at time  $t$ , and  $w_t^i$  is a positive weight given by the probability density at position  $\mathbf{x}_t^i$ , and such that  $\sum_i w_t^i = 1$ . The two steps of Bayesian localization are computed over the particles of eq. 19, which are resampled spatially at each stage to concentrate in regions where the probability density has been found to be higher.

Simple observation models  $p(r|\mathbf{x})$  for Bayesian localization can be produced from range-only path-loss models like the one described in eq. 12, but more accurate models can be obtained with Gaussian Processes (GPs) [39]. GP are a non-parametric, probabilistic approach to function regression which assume that the measurement  $r_i$  follows a normal distribution given as:

$$p(r_i|\mathbf{x}) = N(\rho_{ri}(\mathbf{x}), \sigma_{ri}^2(\mathbf{x})), \quad (20)$$

whose mean and variance are dependent of the position  $\mathbf{x}$ . Linear regression estimations for this distribution are produced from a training set of locations and measured signal strengths  $\{(\mathbf{x}_i, r_{ij})\}$ . The GP approximation has the advantage of being able to handle the nonlinear functional dependency of signal strength and position using relatively simple mathematics, and give sensible estimates for  $p(r_i|\mathbf{x})$  even in regions where calibration data is not available (in this aspect, they are superior to fingerprint techniques). Gaussian processes have been applied successfully to Wifi-based indoor and GSM-based outdoor localization [39] [40], showing accuracy comparable to fingerprinting techniques, with less calibration effort.

## VI. CONCLUSIONS

In this communication we have presented a short review of the mathematical methods commonly employed for indoor

local positioning systems. Localization techniques are divided into four categories: geometry-based methods, minimization of the cost function, fingerprinting, and Bayesian techniques. Their feasibility, implementation requirements, and robustness against outlier noise and occurrence of non line-of-sight propagation of signals are discussed, with key references to the abundant literature already existing in the field of indoor localization. It is hoped that this document might be useful as a first reference for those researchers starting out in this area.

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